# RECTIFIABILITY OF SINGULAR SETS FOR GEOMETRIC FLOWS (I)–YANG-MILLS FLOW

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ABSTRACT. We prove that monotonicity of density and energy inequality imply the rectifiability of the singular sets for Yang-Mills flow.

## 1. INTRODUCTION

In this serial of papers we study the properties of singular sets for geometric flows, such as harmonic map heat flow, Yang-Mills flow, mean curvature flow and Ricci flow. Particularly we are interested in the rectifiability of singular sets and the structure of limit flows for these geometric flows. Generally, we can not expect a global smooth solution to these flows. In many cases it has been proved that these geometric flows may develop singularity as $t$ increasing (see [9][13][17][19]). So to understand the properties and structure of the singularity is a very important problem of the regularity theory of theses geometric flows.

A common property of these geometric flows is the monotonicity of density function. A remarkable result of these papers is that the monotonicity of density and energy inequality imply the rectifiability of the singular sets. Furthermore if the monotonicity formula for density is equality, then all tangent flows of a mean convex mean curvature flow are the dilation of $\mathbb{R}^n$. This is the best possible result for the tangent flows (see [8][18]). The rectifiability and structure of singular sets for harmonic map heat flow was considered in [19].

It is well-known that monotonicity methods are often used in the regularity problems for minimal surfaces, harmonic maps and Navier-Stokes equations, as well as other nonlinear partial differential equations. The monotonicity of density for Yang-Mills flow in static case was first proved by [14]. The monotonicity inequality for a parabolic version of density function, and $\epsilon$-regularity of a solution to Yang-Mills flow were proved in [4][7][13][17]. Hamilton [7] proved monotonicity formulas on manifolds for smooth parabolic flows, including Yang-Mills flow, mean curvature flow and harmonic heat flow.

In this paper we only consider the Yang-Mills flow. The rectifiability of first time singular set for Yang-Mills flow were proved in [17]. Moser [12] considered the rectifiability of singular sets for a class of static solutions including static solutions of Yang-Mills equations.

## 2. Yang-Mills flow

Yang-Mills equations as a field theory were originally arisen in physics. Here we consider the Yang-Mills flow–a parabolic version of Yang-Mills equations. The Yang-Mills flow was first proposed by Atiyah-Bott[2], and it played an important role in Donaldson's four dimensional topology theory [5].

We consider the Yang-Mills flow:

$$\partial_\tau A = -\nabla^* F, \quad \text{in} \quad \mathbb{R}^m \times [0, T)$$
$$A(x, 0) = A_0(x), \quad \text{in} \quad \mathbb{R}^m$$

(2.1)

where $A(x, \tau) = (A_1(x, \tau), A_2(x, \tau), \ldots, A_m(x, \tau))$ is a gauge potential, and for a fixed compact Lie group $G$, $A_\mu$ ($\mu = 1, 2, \ldots, m$) take values in Lie algebra $Lie(G)$. The readers may see [1][17] for the explanation of the notations. However it is not necessary to read [1][17] for understanding the proofs of this paper. In fact, it is enough to consider following special situation.

For simplicity, in this paper we may assume

$$G = U(n) := \{b \text{ is an } n \times n \text{ complex matrix with} : \quad bb^* = I\},$$

where $b^*$ denotes the conjugate transpose of $b$, and $I$ is the unit matrix. The Lie algebra $Lie(U(n))$ can be expressed as

$$Lie(U(n)) = \{B \text{ is an } m \times m \text{ complex matrix with} : \quad B + B^* = 0\}.$$

To define an inner product, we need the following Killing form on $Lie(U(n))$ defined by

(2.2)     $< B, C >= (-1)\text{trace}(BC), \quad \forall B, \ C \in Lie(U(n)).$

Here $\text{trace}(BC)$ means the trace of the matrix $BC$. Furthermore, if

$$B(x) = (B_1(x), B_2(x), \ldots, B_k(x) \quad \text{and} \quad C(x) = (C_1(x), C_2(x), \ldots, C_k(x)),$$

and $B_\mu(x)$ and $C_\mu(x)$ ($\mu = 1, 2, \ldots, k$) are $Lie(U(n))$-valued functions, we define the inner product
(2.3)

$$(B, C) = \int_{\mathbb{R}^m} \sum_{1 \le \mu \le k} < B_\mu(x), C_\mu(x) > dx = \int_{\mathbb{R}^m} \sum_{1 \le \mu \le k} (-1)\text{trace}(B_\mu(x)C_\mu(x))dx.$$

Associated with the potential $A$, the operator $\nabla$ and the gauge field $F$ are defined by

$$\nabla_\mu := \frac{\partial}{\partial x_\mu} + A_\mu$$

and

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} + [A_\mu, A_\nu]$$

respectively. The operator $\nabla^*$ in (2.1) is the adjoint operator of $\nabla$ with respect to the inner product (2.3).

The Yang-Mills flow (2.1) can be regarded as a gradient flow of an action. The invariant action of the Yang-Mills flow is defined by

$$YM(A)(\tau) = \int_{\mathbb{R}^m} |F(x,\tau)|^2 dx, \quad |F(x,\tau)|^2 := \sum_{1 \le \mu,\nu \le m} (-1)\mathrm{trace}(F_{\mu\nu}(x,\tau)F_{\mu\nu}(x,\tau)).$$

A parabolic version of density function to (2.1) is defined by [4][7][13]

$$\theta^\rho(z,\tau) = \rho^{4-m} \int_{\mathbb{R}^m} \exp\{-\frac{|z-y|^2}{4\rho^2}\}|F(y,\tau-\rho^2)|^2 dy, \quad \forall \rho > 0.$$

The monotonicity of density function for static Yang-Mills flow was first proved by [14]. For the parabolic version of density function, the monotonicity inequality

(2.4) $$\theta^\rho(z,\tau) \le C_0 \theta^{\rho'}(z,\tau) + C_1, \quad 0 < \forall \rho \le \forall \rho',$$

and $\epsilon$-regularity of a smooth solution to (2.1) were proved in [4][7][13].

The $\epsilon$-regularity means that there is a constant $\epsilon > 0$ such that $\theta^\rho(z,\tau) < \epsilon$ for some $\rho > 0$ implies that $(z,\tau)$ is a regular point of $A$. So we can define the singular set of the Yang-Mills flow as the complement of the set of the regular points.

**Definition 2.1.** For $\tau > 0$, the singular set of the Yang-Mills flow (2.1) is defined by

(2.5) $$S(\tau) := \{z \in \mathbb{R}^m : \quad \liminf_\rho \theta^\rho(z,\tau) \ge \epsilon\}.$$

Notice that from [13][10] etc., the singular set $S(\tau)$ is not empty for some time $\tau > 0$, and the $(m-4)$-dimension Hausdorff measure of $S(\tau)$ is finite. In this paper, we shall prove the rectifiability of $S(\tau)$.

**Theorem 2.2.** *If $A(x,\tau)$ is a weak solution of (2.1) and its density function has monotonicity, then the singular set $S(\tau)$ of $A(x,\tau)$ at time $\tau > 0$ is $(m-4)$-rectifiable.*

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**Remark 2.3.** (1) The rectifiability for the singular set of steady state Yang-Mills equations is proved by [12], and for the first time singular set of Yang-Mills flow is proved by [17][10].

(2)If $A(x, \tau)$ is a weak solution of (2.1), then as in [19][20], the motion of the singular set $S(\tau)$ can be described by the limit equation of

$$(2.6) \ \ \partial_\tau \theta^\rho(z, \tau) = \Delta_z \theta^\rho(z, \tau) + \frac{1}{\rho^2}(\theta^\rho(z, \tau) - \frac{\rho}{2}\partial_\rho \theta^\rho(z, \tau)), \quad \forall (z, \tau) \in \mathbb{R}^m \times \mathbb{R}^+.$$

The further results on the motion of the singular set may be obtained by considering the properties of solutions to the Yang-Mills flow (see [19]).

(3) Similar results of this paper can be proved on Riemannian manifolds instead of $\mathbb{R}^m$.

## 3. Energy inequality, monotonicity and local continuity of density for Yang-Mills flows

In this section, we shall introduce several basic properties of the Yang-Mills flow, including the energy inequality (Lemma 3.1), the monotonicity of the density function (Lemma 3.2) and local continuity of the density function (Lemma 3.3). The first and second lemma are known fact, and their proofs can be found in the references. Lemma 3.3 was first proved in [19], and for readers' convenience, we give its proof here.

**Lemma 3.1.** *(energy inequality[5][4][13]) Suppose $A$ is a smooth solution of (2.1). Then for all $\tau \in (0, T)$*

$$YM(A)(\tau) + 2 \int_0^\tau \int_{\mathbb{R}^m} |\partial_t A(y, t)|^2 dy dt = YM(A_0).$$

*Particularly we have the energy inequality*

$$(3.1) \qquad\qquad\qquad YM(A)(\tau) \leq YM(A_0).$$

**Lemma 3.2.** *(monotonicity [4][7][13]) Suppose $A$ is a smooth solution of (2.1). Then there is a constant $C > 0$ such that for all $\rho$, $\rho'$: $0 < \rho \leq \rho'$, we have*

$$(3.2) \qquad \theta^\rho(z, \tau) \leq C_0(\rho, \rho')\theta^{\rho'}(z, \tau) + C_1(\rho, \rho'), \quad for \ \ z \in \mathbb{R}^m, \quad \tau > 0,$$

*where*

$$C_0(\rho, \rho') = C \exp\{C(\rho' - \rho)\}, \quad C_1(\rho, \rho') = C(\rho'^2 - \rho^2)YM(A_0).$$

**Lemma 3.3.** *(local continuity) Suppose $A$ is a weak solution of (2.1) which satisfies the energy inequality (3.1) and monotonicity (3.2). For all $R > 1$, for all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}^m$, there is a constant $C$ which is independent of $\rho > 0$ and $x, \tilde{x} \in B_R$, such that*

$$(3.3) \qquad |D_x^\alpha \theta^\rho(\bar{z} + \rho x, \tau) - D_x^\alpha \theta^\rho(\bar{z} + \rho \tilde{x}, \tau)| \leqq C|x - \tilde{x}|.$$

*Proof.* Note that

$$\theta^\rho(\bar{z} + \rho x, \tau) - \theta^\rho(\bar{z} + \rho \tilde{x}, \tau)$$

$$= \frac{1}{2} \int_{\mathbb{R}^m} \{e^{-\frac{|x - \bar{x}|^2}{4}} - e^{-\frac{|\tilde{x} - \bar{x}|^2}{4}}\} |F(\bar{z} + \rho \bar{x}, \tau - \rho^2)|^2 d\bar{x}$$

$$= \frac{1}{2} \int_{|x - \tilde{x}||\bar{x} - x| \leqq r} e^{-\frac{|x - \bar{x}|^2}{4}} \{1 - e^{\frac{|x - \bar{x}|^2 - |\tilde{x} - \bar{x}|^2}{4}}\} |F(\bar{z} + \rho \bar{x}, \tau - \rho^2)|^2 d\bar{x}$$

$$+ \frac{1}{2} \int_{|x - \tilde{x}||\bar{x} - x| \geqq r} e^{-\frac{|x - \bar{x}|^2}{4}} |F(\bar{z} + \rho \bar{x}, \tau - \rho^2)|^2 d\bar{x}$$

$$- \frac{1}{2} \int_{|x - \tilde{x}||\bar{x} - x| \geqq r} e^{-\frac{|\tilde{x} - \bar{x}|^2}{4}} |F(\bar{z} + \rho \bar{x}, \tau - \rho^2)|^2 d\bar{x}.$$

Case 1. $|x - \tilde{x}||\bar{x} - x| \leqq r$ and $|x - \tilde{x}| \leqq \frac{r}{2R}$.
We have

$$|1 - \exp\{\frac{|x - \bar{x}|^2 - |\tilde{x} - \bar{x}|^2}{4}\}| \exp\{-\frac{|x - \bar{x}|^2}{8}\}$$

$$= |1 - \exp\{\frac{(x - \tilde{x}) \cdot (x + \tilde{x}) + 2\bar{x} \cdot (\tilde{x} - x)}{4}\}| \exp\{-\frac{|x - \bar{x}|^2}{8}\}$$

$$(3.4) \qquad \leq \frac{|x - \tilde{x}|(|x + \tilde{x}| + 2|\bar{x}|)}{2} \exp\{-\frac{|x - \bar{x}|^2}{8}\}$$

$$\leq \frac{|x - \tilde{x}|(|x + \tilde{x}| + 2|x| + 2|\bar{x} - x|)}{2} \exp\{-\frac{|x - \bar{x}|^2}{8}\}$$

$$\leq C|x - \tilde{x}|$$

where we used the estimates

$$|x - \tilde{x}|(|x + \tilde{x}| + 2|\bar{x}|) \leqq |x - \tilde{x}||x + \tilde{x}| + 2|x - \tilde{x}||\bar{x} - x| + 2|x - \tilde{x}||x| \leqq 4r$$

and

$$|1 - \exp\{-h\}| \leqq 2h, \quad \text{for} \quad 0 \leqq h \leqq r < 1,$$

because $r$ is small and

$$|\bar{x} - x| \leq \frac{r}{|x - \tilde{x}|}.$$

Case 2. $|x - \tilde{x}||\bar{x} - x| \geqq r$ and $|x - \tilde{x}| \leqq \frac{r}{2R}$.
We have

$$(3.5) \qquad \exp\{-\frac{|x - \bar{x}|^2}{8}\} \leqq \frac{C}{|x - \bar{x}|^2} \leqq \frac{C|x - \tilde{x}|^2}{r^2},$$

and

$$(3.6) \qquad \exp\{-\frac{|\tilde{x} - \bar{x}|^2}{8}\} \leqq \frac{C}{|\tilde{x} - \bar{x}|^2} \leqq \frac{C|x - \tilde{x}|^2}{(r - r^2/4)^2}$$

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where we have used

$$|\bar{x} - \tilde{x}||x - \tilde{x}| \geqq (|\bar{x} - x| - |x - \tilde{x}|)|x - \tilde{x}|$$
$$\geqq |\bar{x} - x||x - \tilde{x}| - |x - \tilde{x}|^2$$
$$\geqq r - \frac{r^2}{4}.$$

Note that

(3.7)
$$\int_{\mathbb{R}^m} e^{-\frac{|x-\bar{x}|^2}{8}} |F(\bar{z} + \rho\bar{x}, \tau - \rho^2)|^2 d\bar{x} \leqq C(YM(A_0) + 1)$$

because if we take

$$\bar{x} := \frac{(\bar{x} - x)}{\sqrt{2}}, \quad \bar{\rho} := \sqrt{2}\rho$$

then

$$\int_{\mathbb{R}^m} e^{-\frac{|x-\bar{x}|^2}{8}} |F(\bar{z} + \rho\bar{x}, \tau - \rho^2)|^2 d\bar{x}$$
$$= 2^{\frac{m-4}{2}} \int_{\mathbb{R}^m} e^{-\frac{|\bar{x}|^2}{4}} |F(\bar{z} + \rho x + \bar{\rho}\bar{x}, \tau + \rho^2 - \bar{\rho}^2)|^2 d\bar{x} \quad \text{(by } x - \bar{x} \to \sqrt{2}\bar{x})$$
$$= 2^{\frac{m-4}{2}} \theta^{\bar{\rho}}(\bar{z} + \rho x, \tau + \rho^2)$$
$$\leq 2^{\frac{m-4}{2}} \{C_0(\bar{\rho}, \sqrt{\tau})\theta^{\bar{\rho}}(\bar{z} + \rho x, \tau + \rho^2) + C_1(\bar{\rho}, \sqrt{\tau})\}, \quad \text{(by monotonicity)}$$
$$\leqq C(YM(A_0) + 1) \quad \text{(by the energy inequality)}.$$

Fix $r$ (small). By (3.4)-(3.7) we obtain (3.3) with $|\alpha| = 0$ in the case of $x, \tilde{x} \in B_R$ and $|x - \tilde{x}| \leqq r/2R$. It is clear for $|x - \tilde{x}| > r/2R$. Similarly, we can prove (3.3) for $|\alpha| > 0$. $\square$

## 4. Rectifiability of singular sets for Yang-Mills flow

For $k = 1, 2, \ldots, m-1$, let $G(m, k)$ denote the set of all $k$-dimensional subspaces passing the origin of $\mathbb{R}^m$.

**Lemma 4.1.** *For $\bar{z} \in \mathbb{R}^m$, we have*

(4.1)
$$\liminf_{\rho} \int_U \rho^{-4} \theta^\rho(\bar{z} + Y, \tau) dY < \infty$$

*for $\gamma_{m,4}$-almost all $U \in G(m, 4)$. Here $\gamma_{m,4}$ denotes the measure on $G(m, 4)$ from the decomposition of the Lebesgue measure $dz = d\gamma_{m,4}dY$ on $\mathbb{R}^m$ (see [11] chapter 3).*

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*Proof.* From Fubini's Theorem and Lebesgue-Fatou lemma,

$$\liminf_{\rho} \rho^{-4} \int_{\mathbb{R}^m} \theta^\rho(z,\tau)dz$$

$$= \liminf_{\rho} \int_{G(m,4)} d\gamma_{m,4}(U) \int_U \rho^{-4}\theta^\rho(\bar{z}+Y,\tau)dY$$

$$\geqq \int_{G(m,4)} d\gamma_{m,4}(U) \liminf_{\rho} \int_U \rho^{-4}\theta^\rho(\bar{z}+Y,\tau)dY.$$

Note that from the definition of $\theta^\rho(z,\tau)$, for all $\tau > \rho^2$

$$\rho^{-4} \int_{\mathbb{R}^m} \theta^\rho(z,\tau)dz = \rho^{-m} \int_{\mathbb{R}^m} dz \int_{\mathbb{R}^m} \exp\{-\frac{|z-y|^2}{4\rho^2}\}|F(y,\tau-\rho^2)|^2 dy$$

$$= \int_{\mathbb{R}^m} |F(y,\tau-\rho^2)|^2 dy \int_{\mathbb{R}^m} \exp\{-\frac{|x|^2}{4}\}dx \leq CYM(A_0),$$

by the energy inequality (3.1).

Then for $\gamma_{m,4}$- almost all $U \in G(m,4)$,

$$\liminf_{\rho} \int_U \rho^{-4}\theta^\rho(\bar{z}+Y,\tau)dY$$

is finite. □

To prove the rectifiability of $S(\tau)$, we need the notation of a cone $X(\bar{z},V,s)$ around a hyperplane $V$ at $\bar{z} \in S(\tau)$ with cone angle $\cos^{-1} s$. For $\bar{z} \in \mathbb{R}^m$, $0 < s < 1$, $0 < r < \infty$ and $V \in G(m,k)$, we define the cone as

$$X(\bar{z},V,s) := \{z \in \mathbb{R}^m : \quad \text{dist}(z-\bar{z},V) < s|z-\bar{z}|\}$$

and define the restriction of the cone $X(\bar{z},V,s)$ in an $m$-dimensional ball $B^m(\bar{z},r)$ as

$$X(\bar{z},r,V,s) := X(\bar{z},V,s) \cap B(\bar{z},r).$$

Notice that $\bar{z} \notin X(\bar{z},V,s)$.

**Lemma 4.2.** *Let $\bar{z} \in S(\tau)$. If for some $W \in G(m,1)$ and $\delta > 0$,*

(4.2) $$\cap_{0<r<\delta} (S(\tau) \cap X(\bar{z},r,W,s)) \neq \emptyset, \quad 0 < \forall s < 1,$$

*then there is a sequence $\{\rho_i\}_i$ and a constant $C = C(\epsilon) > 0$ depending on $\epsilon$ (see 2.5) ) such that*

(4.3) $$\inf_{t\geq 0} \lim_i \theta^{\rho_i}(\bar{z}+\rho_i t\omega,\tau) \geq C(\epsilon)$$

*where $\omega \in S^{m-1}$ is the direction of the line $W = \{t\omega, \quad \forall t \in \mathbb{R}\}$.*

*Proof.* Fix $\bar{z} \in S(\tau)$. Note that for any $t > 0$ and $h > 0$, if we take $s$ such that

$$s \leq \frac{h/2}{\sqrt{t^2 + (h/2)^2}},$$

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then
$$\cup_{\rho_0 \geq \rho > 0} B^m(\bar{z} + \rho t \omega, \rho h/2) \supset X^+(\bar{z}, r, W, s),$$
here $+$ means the positive part of the cone $X$. Note that if we take $t < 0$, then we can cover the negative part of the cone $X$.

So from (4.2) there is a sequence $\{\rho_i\}_i$ such that for each $i \in \mathbb{N}$,
$$S(\tau) \cap B^m(\bar{z} + \rho_i t \omega, \rho_i h/2) \neq \emptyset.$$
Take $\bar{z}_{\rho_i}$ belonging to this joint set. Then $\bar{z}_{\rho_i} \in S(\tau)$ satisfies

(4.4) $\qquad \bar{z}_{\rho_i} \to \bar{z}, \quad \text{and} \quad B^m(\bar{z} + \rho_i t \omega, \rho_i h) \supset B^m(\bar{z}_{\rho_i}, \rho_i h/2).$

We may suppose that for $\{\rho_i\}_i$ the limit
$$\lim_i \theta^{\rho_i}(\bar{z} + \rho_i x, \tau), \quad \forall x \in \mathbb{R}^m$$
exists (if not, we can take a subsequence).

So we have
$$\int_{B^m(t\omega, h)} \lim_i \theta^{\rho_i}(\bar{z} + \rho_i x, \tau) dx$$

$$= \lim_i \rho_i^{-m} \int_{B^m(\bar{z} + \rho_i t\omega, \rho_i h)} \theta^{\rho_i}(z, \tau) dz \quad (\text{by } z = \bar{z} + \rho_i x)$$

(4.5)
$$\geqq \lim_i \rho_i^{-m} \int_{B^m(\bar{z}_{\rho_i}, \rho_i h/2)} \theta^{\rho_i}(z, \tau) dz \quad (\text{by } (4.4))$$

$$\geqq \inf_{z \in S(\tau)} \int_{B^m(0, h/2)} \liminf_\rho \theta^\rho(z + \rho x, \tau) dx \quad (\text{by } z \to \bar{z}_{\rho_i} + \rho_i x)$$

$$\geq C(\epsilon) h^m, \quad (\text{by } (2.5), \text{ where } C(\epsilon) > 0 \text{ only depends on } \epsilon).$$

Noting the continuity of $\theta^\rho(\bar{z} + \rho x, \tau)$ about $x$ (see Lemma 3.3), the left side of (4.5) is smaller than
$$C h^m \lim_i \theta^{\rho_i}(\bar{z} + \rho_i t\omega, \tau) + C(t) h^{m+1}.$$

If we fix $h$ such that
$$C(\epsilon) - C(t) h \geq C(\epsilon)/2$$
then we get (4.3). $\qquad \square$

**Proposition 4.3.** *If $A(x, \tau)$ is a weak solution of (2.1) which satisfies the energy inequality and its density function satisfies the monotonicity inequality, then the singular set $S(\tau)$ of $A(x, \tau)$ at time $\tau > 0$ is $(m - 4)$- rectifiable.*

*Proof* If $S(\tau)$ is not $(m - 4)$-rectifiable, then there is a purely $(m - 4)$-unrectifiable subset $S$ of $S(\tau)$ such that for any $U \in G(m, 4)$, for any $\delta > 0$

(4.6) $\qquad \limsup_{s \to 0} \sup_{0 < r < \delta} (rs)^{4-m} \mathcal{H}^{m-4}(S \cap X(\bar{z}, r, U, s)) > 0$

for $\mathcal{H}^{m-4}$ almost all $\bar{z} \in S$ (see [11] Corollary 15.15).

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We select special sequences of $\delta$, $s$ and $r$ by the following three steps.

(1) Take a sequence $\{\delta_i\}_{i\in\mathbb{N}}$ such that $\delta_i \downarrow 0$ as $i \uparrow \infty$;

(2) For each $\delta_i$, take a sequence $\{s_j^i\}_{j\in\mathbb{N}}$ such that

$$\text{for each } i \quad s_j^i \downarrow 0, \quad \text{as well as} \quad s_j^j \downarrow 0, \quad \text{as} \quad j \uparrow \infty$$

and

$$\sup_{0<r<\delta_i} (rs_j^i)^{4-m}\mathcal{H}^{m-4}(S\cap X(\bar{z},r,U,s_j^i)) \geqq C(i)(>0, \text{ independent of } j);$$

(3) Furthermore, for each $i$ there is $\{r_j^i\}_{j\in\mathbb{N}}$: $0 < r_j^i < \delta_i$, such that

$$r_j^j \downarrow 0, \quad \text{as} \quad j \uparrow \infty,$$

and for each $j$

$$\mathcal{H}^{m-4}(S\cap X(\bar{z},r_j^i,U,s_j^i)) \geqq \frac{C(i)}{2}(r_j^i s_j^i)^{m-4}.$$

It implies that in case of $i = j = k$ there are $\{z(k) \in S\cap X(\bar{z},r_k^k,U,s_k^k)\}_{k\in\mathbb{N}}$ such that

$$z(k) \to \bar{z}, \quad \text{as} \quad k \to \infty, \quad \text{because } r_k^k \downarrow 0.$$

Let

$$(\omega_1(k),\omega_2(k)) := \frac{z(k) - \bar{z}}{|z(k) - \bar{z}|},$$

where $\omega_2(k) \in S^3 \subset U$ and $\omega_1(k) \to 0$ as $k \to \infty$, because $s_k^k \downarrow 0$. By the compactness of $S^3$, there is a subsequence (denoted also by $\{z(k)\}$) such that

$$\omega_2(k) \to \omega \in S^3 \subset U.$$

Let $W = \{t\omega, \quad \forall t \in \mathbb{R}\}$. Then $W \subset U$ and for any fixed $s \in (0,1)$, there is $k_0 > 0$ such that for $k \geq k_0$,

$$s_k^k \leq s, \quad z(k) \in S\cap X(\bar{z},r_k^k,W,s), \quad z(k) \to \bar{z} \quad (k \to \infty).$$

Thus, there is

$$W \subset U, \quad W \in G(m,1)$$

such that (4.2) is satisfied. So by Lemma 4.2 we have (4.3) at least in one direction of $W$. For any $U$ in $G(m,4)$, from the continuity of $\lim_i \theta^{\rho_i}(\bar{z} + \rho_i x, \tau)$ on $x$, we get a $m$-dimension set of $x$ with infinity $m$-dim Lebesgue measure such that on this set

$$\lim_i \theta^{\rho_i}(\bar{z} + \rho_i x, \tau) \geqq C(>0),$$

which is contradiction with Lemma 4.1 where $\rho$ is replaced by $\rho_i$. $\qquad\square$

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